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Explicit computation of Galois p -groups unramified at p

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Abstract

We introduce a new method for finding Galois groups, by modifying the p -group generation algorithm using number-theoretical information. Thus, by exhaustive search, we calculate the Galois groups of some p -extensions that were previously beyond reach.
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0. Introduction

In this paper we introduce a new method for finding Galois groups by computer. This is particularly effective in the case of Galois groups of p -extensions ramified at finitely many primes but unramified at the primes above p . Such Galois groups have been regarded as amongst the most mysterious objects in number theory [12]. Very little has hitherto been discovered regarding them despite their importance in studying p -adic Galois representations unramified at p . The conjectures of Fontaine–Mazur [8] say that they should have no p -adic analytic

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quotients (equivalently, the images of the Galois representations should always be finite), and there are generalizations due to the first author [3,4], suggesting that they should instead have ‘large’ actions on certain trees.

Below, we indicate how the method works in the case of 2-extensions of \mathbf{Q} . This case is chosen since powers of 2 grow relatively slowly allowing us extensive computations. The idea is to modify the p -group generation algorithm [11] so that as it goes along, it uses number-theoretical information to eliminate groups that cannot arise as suitable quotients of the Galois group under investigation. In the best cases we obtain a short list of candidates for the Galois group. Two phenomena are worth noting. First, as noted in [5], the same groups turn up repeatedly. Second, even when the short list consists of more than one group, the candidates typically are so similar that most questions regarding them will yield the same answer. In this way, we can tell a lot about the Galois groups without always pinning them down exactly.

Two items of progress are worth noting here. First, our exhaustive search allows us to go well beyond what was known previously, which was in general only the maximal nilpotency class 2 quotient of these Galois groups (see, e.g., [9]). Second, we obtain at least a conjectural characterization of the Galois group of the maximal 2-extension unramified outside $\{p, q\}$ in most cases when $p \equiv 3 \pmod{4}$ and $q \equiv 5 \pmod{8}$.

Recently, our method was modified to find the Galois groups of p -class towers of certain quadratic fields [6], including resolving an old question of Stark. Also, Koch [10] has shown that the hypotheses of some of our examples hold for all pairs of primes p, q in congruence classes modulo small powers of 2.

1. Some basics

Let S be a finite set of odd primes. The maximal 2-extension of \mathbf{Q} unramified outside S (allowing ramification at infinity) will be denoted by \mathbf{Q}_S and its Galois group over \mathbf{Q} by G_S . We seek to find G_S given a set S . We begin with some elementary results.

Theorem 1. (1) *For every finite index subgroup H of G_S , H/H' is finite.*

(2) [9, Section 4] *The generator rank $d(G_S)$ equals the relator rank $r(G_S)$. Equivalently, G_S has trivial Schur multiplier.*

Proof. (1) follows by class field theory, from the finiteness of certain ray class groups. (2) is a theorem of Shafarevich. \square

Corollary. (1) *If $\#S = 1$, then G_S is a finite cyclic group.*

(2) *If $\#S \geq 4$, then G_S is infinite (Golod–Shafarevich).*

In [5], the first author and Perry exploited various circumstances in which a decomposition subgroup has index 1 or 2 in G_S . There it was noted that there is a tendency for G_S to belong to a small family of possibilities. This paper arose from the desire to study cases where G_S does not have a decomposition subgroup of small index. (It can be checked that the groups given below do not have any metacyclic subgroups of small index. Note that decomposition subgroups must be metacyclic here since all ramification is tame.)

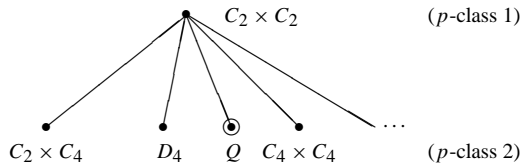
2. The method

The idea is to modify O'Brien's p -group generation algorithm [11], which allows computer algebra systems such as MAGMA [2] to find systematically all p -groups with some property; e.g. all 2-groups with 2 generators and order ≤ 1024 . Our modification is to have the program save those descendants that satisfy various conditions coming out of the number theory.

O'Brien's approach is to define, given p -group G , a sequence of characteristic subgroups by $P_1(G) = G$, $P_i(G) = P_{i-1}(G)^p(G, P_{i-1}(G))$. This is called the lower p -central series. Thus, $G = P_1(G) \geq P_2(G) \geq \dots$. The smallest c such that $P_{c+1}(G) = \{1\}$ is called the p -class of G . Since the i th term $\gamma_i(G)$ of the lower central series of a p -group G is contained in the i th term $P_i(G)$ of the lower p -central series, the p -class is bounded below by the nilpotency class.

A p -group H is called a *descendant* of G if $H/P_{c+1}(H) \cong G$, where c is the p -class of G . It is an *immediate descendant* if it has p -class $c + 1$.

O'Brien's algorithm finds all immediate descendants of a given p -group G . These can be conveniently arranged in the form of a tree. For instance, the immediate descendants of $C_2 \times C_2$ look like in the following diagram:



This can then be iterated to give all p -class 3, then p -class 4, etc., descendants of $C_2 \times C_2$. The tree contains each 2-group with 2 generators exactly once (up to isomorphism). The circle by the quaternion group, Q_8 , indicates that it is 'terminal', i.e. has no descendants. The infinite ends of the tree yield infinite pro-2 groups; for instance $C_2 \times C_2 \leftarrow D_4 \leftarrow D_8 \leftarrow \dots$ has inverse limit the dihedral pro-2 group.

Our aim then is, given S , c , to compute a short list (hopefully just one group) of possibilities for the p -class c quotient $G_S/P_{c+1}(G_S)$.

We impose three kinds of number-theoretical constraints on G_S :

- (I) (Top-end information) We can compute H/H' for H of small index in G_S (index ≤ 2 by class field theory, ≤ 16 by machine, e.g. KASH [7]). The point is that H/H' is isomorphic to the 2-part of a ray class group of the corresponding field.
- (II) (Local information) We know that G_S has inertial generators $\{\tau_p: p \in S\}$ with the property that τ_p is conjugate to τ_p^p [9]. It also contains a complex conjugation τ_∞ of order 2.
- (III) (Multiplier information) $r(G_S) = d(G_S)$.

Note that if we had been considering p -extensions ramified at p , then H/H' need not have been finite, which would make our approach much less effective.

3. An example

The first case not covered by the explicit families above is that of $S = \{5, 19\}$. We attempt to compute G_S by O'Brien's program using the number-theoretical constraints above.

First, we show that $G_S/P_4(G_S)$ is a certain group P of order 64. This is done by showing that $x^{16} - 70x^{14} + 1161x^{12} + 5125x^{10} + 8910x^8 + 7783x^6 + 5235x^4 + 593x^2 + 25$ has P as Galois group, that its root field is unramified outside S , and that P has 2-class 3 and is not a quotient of any other 2-class 3 2-group. All these statements are established by commands in MAGMA [2]. The polynomial itself can be found by KASH [7], which has a command for computing quadratic ray class extensions.

Second, we find the conjugacy classes of P where $\tau_5, \tau_{19}, \tau_\infty$ can lie.

Third, we use KASH to compute H/H' for all subgroups H of index ≤ 4 in G_S . KASH has a command for finding ray class groups of number fields. We compute the small degree subfields of the fixed field of P and then the Sylow 2-subgroups of their ray class groups.

All the above goes in an input file. Our program now finds all immediate descendants of P and tests them for 3 properties:

- (i) that they contain elements x_5, x_{19} that are respectively conjugate to their 5th and 19th powers and x_∞ of order 2 such that their images in P are in the conjugacy classes found above,
- (ii) that the abelianizations of their subgroups of index ≤ 4 are smaller than (i.e. quotients of) the abelianizations of the corresponding subgroups of G_S ,
- (iii) that their nuclear rank differs from their 2-multiplier rank by at most 2 (see [11] for definitions of these terms).

The last condition arises from the following lemma.

Lemma. *Let G be a d -generated pro- p -group (possibly finite) with trivial Schur multiplier. Let $n > 1$. Then the nuclear rank and p -multiplier rank of $G/P_n(G)$ differ by at most d .*

Proof. Let H be the p -covering group of $G/P_n(G)$. Then $G/P_{n+1}(G)$ is obtained from H by dividing out by a supplement N to the nucleus of $G/P_n(G)$. Consider the fiber product F of G and H over $G/P_{n+1}(G)$. This group maps onto G , with kernel N . Now F is a central extension of G by N , and $N \leq P_2(G)$ as $n \geq 2$. Thus N is isomorphic to a subgroup of the p -multiplier $H_2(G, \mathbb{Z}/p\mathbb{Z})$ of G . But

$$H_2(G, \mathbb{Z}/p\mathbb{Z}) \cong H_2(G, \mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z} \oplus \text{Ext}(G/G', \mathbb{Z}/p\mathbb{Z})$$

by the Künneth formula; and hence is elementary abelian of rank d , since the Schur multiplier of G is trivial. Thus N has rank at most d . \square

Each group that passes all three tests is then saved, its immediate descendants computed, and the tests repeated on these. The search is done by depth first so as to save on storage. For large enough 2-class (ii) can be refined, since at this depth the abelianizations must actually equal those of the corresponding subgroups of G_S .

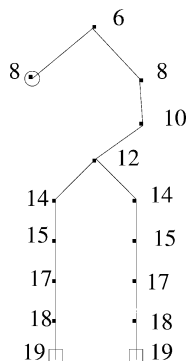
This continues until we reach groups with abelianizations of their subgroups of index ≤ 4 exactly right and with trivial Schur multiplier (which implies that the group is terminal [11]). These groups are then candidates for G_S .

If the process terminates, then we know that G_S is finite and is one of the candidate groups. It might happen, however, that the process does not terminate in a reasonable length of time. In that case, it could be that there are infinite groups satisfying our number-theoretical constraints or that there are only finite groups but their p -class is so large that they have yet to be reached. The input file to our program contains a limit for the p -class, which can be adjusted. In the case of $\{5, 19\}$, the process terminates in groups of 2-class 11.

Fig. 1 shows what the tree for $S = \{5, 19\}$ looks like. The numbers next to the nodes give the exponents of the orders. A circle around a node means that none of the group's descendants pass all three tests and that the group itself cannot be G_S . A square around a node means that we have reached a group that satisfies everything G_S is known to satisfy.

Theorem 2. *If $S = \{5, 19\}$, then G_S is of order 2^{19} and 2-class (indeed nilpotency class) 11. In fact, it is isomorphic to $\langle a, b: a^2, babab^3ab^{-2}ab^{-2}ab^{-5}a \rangle$ or to $\langle a, b: a^2, bab^2ab^2abab^3ab^{-5}a \rangle$.*

These two groups are very similar to each other. For instance, their lattices of subgroups of index ≤ 32 (even with abelianizations attached) match. This makes them indistinguishable by top-end methods; i.e. by calculation in low degree subfields of \mathbf{Q}_S . The advantage of this, though, is that the answer to many

Fig. 1. The tree for $S = \{5, 19\}$.

questions about G_S can be given, since it is the same for either group. One can prove that in the sense of lengths of relations the given presentations are the simplest possible for the groups (by exhaustively checking all shorter relations; i.e. that no group $\langle a, b: a^2, r \rangle$ with r a shorter word in a, b is isomorphic to either to them).

Remarks. (1) The largest metacyclic subgroup of G_S has order 2^9 and so decomposition subgroups have index at least 2^{10} . This means that the methods used in [5] for the earlier explicit families simply do not carry over.

(2) In fact, if we only use abelianizations for subgroups of index ≤ 2 (which can be computed by class field theory rather than by machine), then we get the same tree and the same two groups output.

(3) Several other choices of S lead to the same tree and indeed the same pair of groups, e.g., $\{3, 37\}$, $\{3, 61\}$, $\{5, 59\}$, $\{5, 139\}$, $\{11, 53\}$, \dots .

Next we compute some other trees. For $S = \{13, 23\}$, we get the tree in Fig. 2. The same tree and groups are output for $S = \{13, 103\}$, $\{5, 199\}$ and many other choices of S .

Theorem 3. *If $S = \{13, 23\}$, $\{13, 103\}$, $\{5, 199\}$, \dots , then G_S is of order 2^{24} and 2-class (indeed nilpotency class) 15.*

The tree for $S = \{5, 79\}$ is as in Fig. 3.

Theorem 4. *If $S = \{5, 79\}$, then G_S is of order 2^{29} and 2-class (indeed nilpotency class) 19.*

It should be noted that no short presentations of these groups have been found. These trees and their output groups seem to fit into a pattern that we now

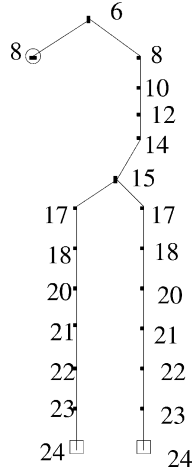


Fig. 2. The tree for $S = \{13, 23\}$.

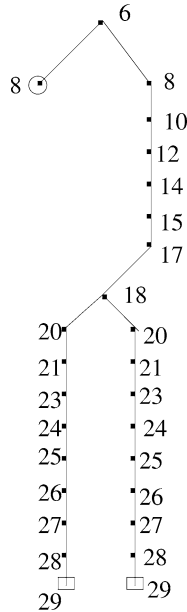


Fig. 3. The tree for $S = \{5, 79\}$.

formalize as a conjecture. (This pattern has been checked experimentally as far as $S = \{13, 127\}$, i.e. $k = 7$. Note, however, that we take the further step, which we cannot check, of proposing that of the two candidate groups that always arise, one is never actually the Galois group.)

First, note that by class field theory, if $p \equiv 3 \pmod{4}$, $q \equiv 5 \pmod{8}$, and p is a square but not a 4th power modulo q , then the abelianizations of the subgroups of index 2 are $[4, 4]$, $[2, 2, 2]$, $[2, 2^n]$ ($n \geq 4$), corresponding to $\mathbf{Q}(\sqrt{-p})$, $\mathbf{Q}(\sqrt{q})$, $\mathbf{Q}(\sqrt{-pq})$, respectively.

Suppose that $n = 4$ (as it does in all the cases above).

Conjecture. *There is a family of groups $G(k)$ ($k \geq 2$) such that $G(k)$ has order 2^{5k+9} and nilpotency class $4k + 3$, and such that if 2^k exactly divides $p + 1$ (together with the other assumptions above), then $G_S \cong G(k)$.*

The main problem is to identify the groups $G(k)$. There are some similarities such as those noted above. Also, the number of generators of their normal subgroups apparently never exceeds 4 and there is often a unique maximal abelian normal subgroup $A(k)$ of order 2^{3k+6} . On the other hand, the structures of $G(k)'/G(k)''$ and $A(k)$ vary erratically. For $k = 2, 3, 4, 5$, the abelian invariants of $G(k)'/G(k)''$ are respectively $[2, 4, 16]$, $[8, 8, 8]$, $[4, 16, 32]$, $[4, 32, 64]$, whereas those of $A(k)$ are respectively $[4, 4, 8, 32]$, $[4, 8, 32, 32]$, $[4, 16, 64, 64]$, $[4, 16, 256, 256]$. Note that for $k = 5$, $A(k)$ has order 2^{22} rather than the predicted 2^{21} .

4. The case $S = \{p, q, r\}$

We now produce some cases with $\#S = 3$ and G_S infinite. The Fontaine–Mazur conjecture and its generalizations apply to this case. Our method yields large quotients of G_S against which to test out these conjectures.

Suppose $p, q, r \equiv 3 \pmod{4}$. This keeps G_S/G'_S small, namely $C_2 \times C_2 \times C_2$. The theory now comes in two flavors according as

- (a) $\{p, q, r\}$ can be ordered so that $\left(\frac{p}{q}\right) = \left(\frac{q}{r}\right) = \left(\frac{r}{p}\right)$ or
- (b) they cannot.

4.1. Case (a)

In this case, G_S is apparently always finite. Consider, for example, $S = \{3, 11, 19\}$. The abelianizations for the seven subgroups of index 2 consist of $[4, 4]$ once, $[2, 2, 4]$ three times, and $[2, 2, 8]$ three times. Taking as input the 2-class 1 quotient $P = G_S/P_2(G_S) \cong C_2 \times C_2 \times C_2$, our method yields the tree in Fig. 4.

Theorem 5. *If $S = \{3, 11, 19\}$, then G_S is of order 2^{14} and 2-class (indeed nilpotency class) 5.*

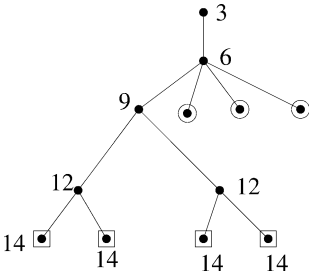


Fig. 4. The tree for $S = \{3, 11, 19\}$.

Once again, we are not able to tell which of the four groups output is actually G_S , but once again it does not matter for many purposes since the answer to many questions about them is the same.

4.2. Case (b)

In this case, G_S is always infinite. This is because a subgroup of index 2 has abelianization $[2, 2, 2, 2]$ and has as fixed field an imaginary quadratic field and then we apply the result of Shafarevich that such a subgroup must have 4 generators and 4 relations and hence be infinite by Golod–Shafarevich.

Let, e.g., $S = \{3, 19, 43\}$. In this case the abelianizations of the index 2 subgroups are as small as they can be, namely $[4, 4]$ once, $[2, 2, 4]$ three times, and $[2, 2, 2, 2]$ three times. If we start with the 2-class 2 quotient $P = G_S/P_3(G_S)$ of order 64, then we get a tree as in Fig. 5.

4.3. Comments

4.3.1. The method has been used successfully in the case of 3-groups. For instance, in the case of three primes $S = \{p, q, r\}$ that are 1 (mod 3) but not 1 (mod 9), the first author has computed many cases of the Galois group of the maximal 3-extension unramified outside S . In each case, we obtain a finite group, in agreement with the result of Andozskii–Cvetkov [1], which says that this group is either finite or isomorphic to $\Gamma := \ker(SL(2, \mathbf{Z}_3) \rightarrow SL(2, \mathbf{F}_3))$. The latter

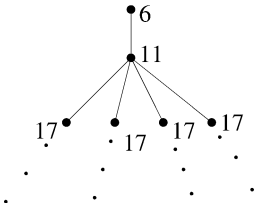


Fig. 5. The tree for $S = \{3, 19, 43\}$.

would contradict the Fontaine–Mazur conjecture. This situation has been explored in greater detail by the first author and Farshid Hajir.

4.3.2. The Andozskii–Cvetkov groups have the interesting property that they form a family of 3-groups (in fact with 3 replaced by any odd prime) $G := G(n)$ such that $G/P_n(G)$ is isomorphic to $\Gamma/P_n(\Gamma)$ where Γ is the above congruence subgroup. This suggests that in general the groups arising might fall into families with $G/P_n(G)$ isomorphic to $H/P_n(H)$ for some infinite ‘governing’ group H . For instance, for our family in [5], such a role is played by $H = \langle a, b: a^2 = b^{-1}ababab^{-1}a = 1 \rangle$ (we simply take the 2-adic limit of the given relation). This is the 2-adic space group $\mathbf{Z}_2 \wr C_2$. Unfortunately, there is no obvious candidate for such a governing group for our family in Section 3.

It is also unfortunate that Γ satisfies the three key properties we use to find our Galois groups. In other words, there exists S such that the top-end information matches that of Γ , and it is easy to see that Γ is generated by elements x such that x is conjugate to x^n for some $n > 1$ and that $d(\Gamma) = r(\Gamma) = 3$. Perhaps there is some additional piece of information we have overlooked that excludes Γ from being one of our Galois groups (in accord with the Fontaine–Mazur conjecture).

4.3.3. As Hajir has noted, G_S can be infinite even if $\#S = 2$; for instance, if $S = \{17, 101\}$. This follows by applying Golod–Shafarevich to the maximal real subfield of $\mathbf{Q}(\zeta_{17})$. This introduces the interesting possibility of constructing a family of examples for which the first few are finite and can be found explicitly but the later ones are infinite. For instance, $\{17, 101\}$ might be in a family with general term $\{p, 6p - 1\}$ with 2^k exactly dividing $p - 1$. The first term is, e.g., $S = \{5, 29\}$ and we intend to compute G_S once we have sufficient computational power. Replacing the 4 by 16 (somehow) in the presentation yields a guess for G_S when $S = \{17, 101\}$.

5. Impact on the Fontaine–Mazur conjecture and its generalizations

Let W be the automorphism group of the binary rooted tree. In other words, we set $W_1 = C_2$ and $W_n = W_{n-1} \wr C_2$, so that W_n is isomorphic to the Sylow 2-subgroup of the symmetric group on 2^n letters. The exponent of its order is $2^n - 1$. Let $W = \lim_{\leftarrow} W_n$.

If G is a closed subgroup of W , its Hausdorff dimension is defined to be

$$\liminf_{n \rightarrow \infty} \frac{\log |G_n|}{\log |W_n|},$$

where G_n is the image of G in W_n . The first author has made the following conjecture regarding G_S [4]. (Note that a just-infinite pro- p group is one that is infinite but all of its proper quotients are finite.)

Conjecture. *The just-infinite quotients of G_S are isomorphic to closed subgroups of W of nonzero Hausdorff dimension.*

Note that this implies the Fontaine–Mazur conjecture in this case, in that a 2-adic analytic group cannot embed in W with nonzero Hausdorff dimension. (The conjecture amounts to saying that the just-infinite quotients of G_S are Grigorchuk’s branch groups, together with a conjectural characterization of branch groups in terms of Hausdorff dimension.)

The conjecture also implies that if G_S is infinite, then there exists a representation $G_S \rightarrow W$ with image of nonzero Hausdorff dimension. We use our new computational tools to provide limited evidence to back this up, by exhibiting a representation $G_S \rightarrow W_4$ with relatively large image.

Example. Let $S = \{3, 19, 43\}$. Let

$$f = x^{16} - 4x^{15} + 10x^{13} + 14x^{12} - 42x^{11} - 20x^{10} + 80x^9 + 15x^8 \\ - 100x^7 + 10x^6 + 78x^5 - 25x^4 - 26x^3 + 9x^2 - 10x + 13.$$

Its Galois group is a 2-group since its root field has a chain of subfields of every index. The splitting of the polynomial modulo various primes tells us by van der Waerden’s method that its Galois group contains elements with various cycle structures. Using the MAGMA database of transitive groups of degree 16, we find there is exactly one subgroup of W_4 that contains elements with all these cycle structures and that has abelianization $C_2 \times C_2 \times C_2$. This then gives us a map from $G_S \rightarrow W_4$ with image $\text{TransitiveGroup}(16, 1735)$ (i.e. the 1735th transitive permutation group of degree 16 in the MAGMA database) of order 2^{13} .

Acknowledgments

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References

- [1] I.V. Andozskii, V.M. Cvetkov, A certain series of finite closed p -groups, *Izv. Akad. Nauk SSSR Ser. Mat.* 38 (1974) 278–290 (in Russian).
- [2] W. Bosma, J. Cannon, *Handbook of MAGMA Functions*, School of Mathematics and Statistics, University of Sydney, 1993.
- [3] N. Boston, Some cases of the Fontaine–Mazur conjecture II, *J. Number Theory* 75 (1999) 161–169.
- [4] N. Boston, Tree representations of Galois groups, preprint, see <http://www.math.uiuc.edu/Algebraic-Number-Theory/0259/index.html>.

- [5] N. Boston, D. Perry, Maximal 2-extensions with restricted ramification, *J. Algebra* 232 (2) (2000) 664–672.
- [6] M.R. Bush, Computation of Galois groups associated to 2-class towers of some quadratic fields, preprint, see <http://www.math.uiuc.edu/Algebraic-Number-Theory/0302/index.html>.
- [7] M. Daberkow, C. Fieker, J. Klüners, M. Pohst, K. Roegner, M. Schörmig, K. Wildanger, KANT V4, in: *Computational Algebra and Number Theory*, London, 1993; *J. Symbolic Comput.* 24 (3–4) (1997) 267–283.
- [8] J.-M. Fontaine, B. Mazur, Geometric Galois representations, in: *Elliptic Curves and Modular Forms*, Proceedings of a conference held in Hong Kong, December 18–21, 1993, International Press, Cambridge, MA, and Hong Kong, 1993.
- [9] A. Fröhlich, Central extensions, Galois groups, and ideal class groups of number fields, in: *Contemp. Math.*, Vol. 24, AMS, 1983.
- [10] H. Koch, On maximal 2-extensions of \mathbf{Q} with given ramification, preprint.
- [11] E.A. O'Brien, The p -group generation algorithm, *J. Symbolic Comput.* 9 (1990) 677–698.
- [12] K. Wingberg, On the maximal unramified p -extension of an algebraic number field, *J. Reine Angew. Math.* 440 (1993) 129–156.